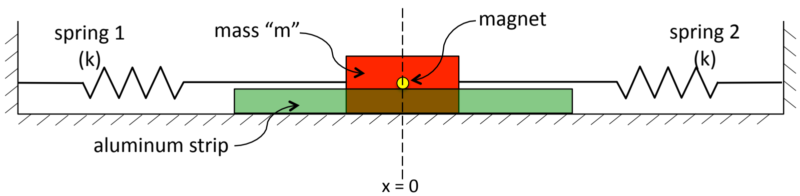
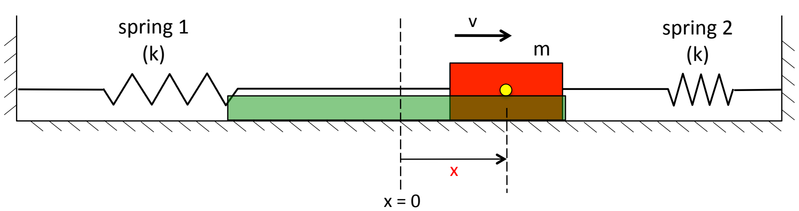
THE PROBLEM

along with some Preliminary Information

There are times when analyzing a situation is most easily done using straight math, but there are also times when that is horrifically difficult and being "clever" works better. What you are about to experience is one of those clever approaches. Here's the set-up. A mass with an attached magnet sits on a frictional surface. It has springs attached to each end and is initially in a state of static equilibrium. An aluminum strip is positioned along the mass's path. The situation is shown below:

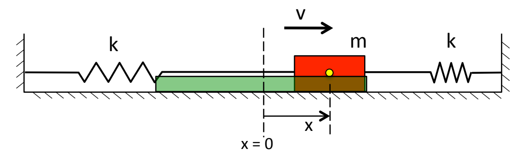


Here is the problem: If the mass is off-set by some distance *x* and released, *how does the mass's velocity act over time?* The dynamic situation is shown below.



What follows is a rundown of the physics involved with this situation:

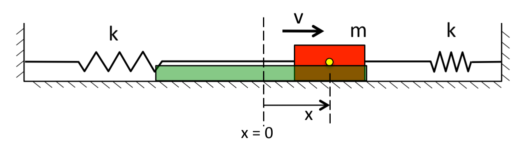
1.) According to Newton's Second Law, the net force acting on a body in a particular direction (in this case, in the "x" direction) will be proportional to the acceleration of the body, where the proportionality constant between the two quantities is the mass "m" of the body. In other words,



$$F_{net,x} = ma_x$$

2.) Dropping the subscript (everything is happening in one dimension), the forces in play on the mass along the direction of motion will be generated by the springs and by the drag force produced by the eddy currents created as the magnet and aluminum strip interact. To that end:

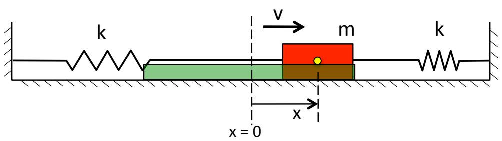
3.) The spring force produced by an ideal spring is proportional to the displacement x of the spring (relative to the equilibrium position at x = 0). To make this into an equality, we have to multiply by the spring's spring constant k (the *spring constant* essentially measures the stiffness of the spring). That is:



$$F_{spring} = -kx$$

(Minor note: the negative sign is needed because when the displacement is to the left with an x-coordinate being negative, the force is back toward equilibrium to the right, which is in the positive direction. For the right and left side of the equation to be equal, the effective signs have to match up, hence the need for the added negative sign.)

4.) The damping force produced by the eddy currents in the aluminum strip is proportional to the velocity of the mass. Taking the velocity to be v, the proportionality constant needed to make these two parameters into an equality is called the *damping constant*. For our purposes, we will symbolize it with a D. As such, we can write:



$$F_{damping} = -Dv$$

(Again, a minor note: the negative sign here is required as the damping force will always be in a direction *opposite* the direction of the velocity vector.)

5.) A quick rundown of Calculus:

a.) The derivative of a function gives you a second function that identifies the rate at which the original function *changes*.

i.) On a graph, the evaluation of a function's derivative at a point yields a second function that defines the slope of the graph at the point of interest.

ii.) It is perfectly possible to be interested in how a function changes as one moves through space. This is called a *spatial derivative*. They are generally denoted as df/dx.

iii.) It is also reasonable to consider how a function changes in time. These are called *time derivatives*.

iv.) In physics, *time derivatives* can be denotes in several ways. They are:

$$\frac{df}{dt} \quad \text{or} \quad \dot{f} \quad \text{or} \quad f'$$

b.) The second derivative of a function gives you a new function that identifies the rate at which the first derivative changes. In physics, second derivatives in *time* are denoted as:

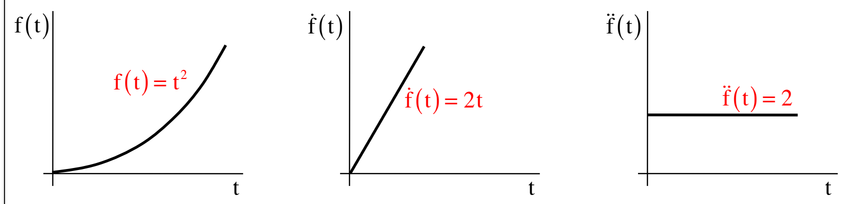
$$\frac{d^2f}{dt^2} \quad \text{or} \quad \ddot{f} \quad \text{or} \quad f''$$

c.) Example:

If the time-dependent function was $f(t) = t^2$,

the time-derivative function (i.e., f(t)'s slope function) is $\dot{f}(t) = 2t$

and its second time-derivative (i.e., the slope of $\dot{f}(t)$) is $\ddot{f}(t) = 2$



d.) When it comes to physical parameters (i.e., variable that pertain to motion):

i.) The first derivative of a position function $x(t)$ yields a velocity function $v(t)$. That is:

$$\dot{x} = \frac{dx}{dt} = v$$

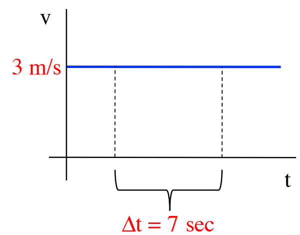
ii.) The first derivative of a velocity function $v(t)$, which is also the second derivative of the position function $x(t)$, yields an acceleration function $a(t)$. That is:

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} \quad \text{or} \quad \dot{v} = \ddot{x} = a$$

iii.) These are all the ways the position, velocity and acceleration functions are related and characterized.

d.) An integral denotes a summation of tiny bits of a whole.

i.) Observation: Let's say an object is moving with constant velocity 3 m/s. How far will it travel in 7 seconds (a graph of the motion is shown to the right).



ii.) A quick use of the old *distance equals rate (velocity) times time* formula yields an answer of 21 meters. Simply and easy.

iii.) What's important to note here is that 21 happens to be the area under the *velocity versus time* graph over a 7 second period.

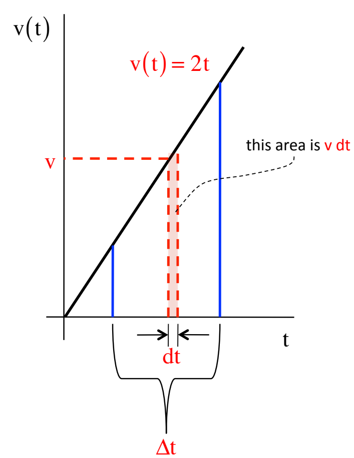
a.) Don't believe me? Look at the graph. The "height" is 3. the "width" is 7, and product of the two is 3x7=21.

THE MORAL? There will be times when the *area under a curve* will have relevance.

d.) (cont'd) So what's the deal with integrals?

i.) Example: Consider an object moving with velocity $v = 2t \ (m/s)$. How far will it travel over a given period of time?

ii.) Again, we know that *distance equals rate (i.e., velocity) times time*, so all we have to do is multiply v and Δt and we have it . . . except v isn't a constant here. So what to do?



Solution: We break the time interval into differentially small pieces of time of length dt (this is like Δt, but really, really small) determine the product $v\,dt$ for all of the little pieces, then sum them all up.

iii.) In a situation like this, the summation sign doesn't look like $\sum$, it looks like $\int$.

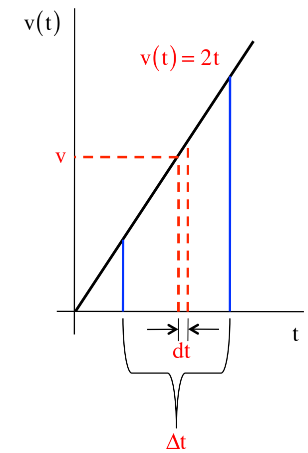
iii.) This summing process is called *integrating*. In its indefinite form it is denoted as:

$$\int v \ dt$$

This process yields a second function $x(t)$ the evaluation of which yields the distance traveled between *any two points in time*.

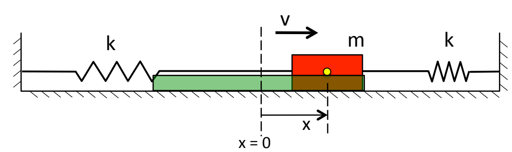
iv.) For our example, then, when $v = 2t \ (m/s)$ we can write:

$$x(t) = \int (2t) \ dt$$
$$= t^2$$



v.) How do we know that the integral of $2t$ equals t^2? Not important here. What's important is that you understand that a derivative is associated with the rate of change of a function, and an integrals is a summing process associated with the area under a graph.

6.) So back to our spring problem and Newton's Second Law.



a.) If we assume the mass is moving to the right so the damping force is negative, summing the forces on the mass yields:

$$\underline{\sum F:}$$

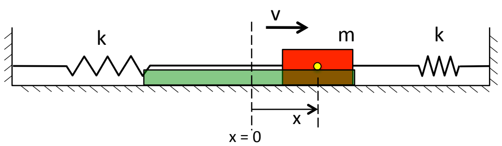
$$F_{spring1} + F_{spring2} + F_{damping} = ma$$
$$\Rightarrow \ [-k(x)] + [-k(x)] + [-D(v)] = ma$$
$$\Rightarrow \ -2kx \quad -D\dot{x} \quad = m\ddot{x}$$
$$\Rightarrow \ \ddot{x} + \left(\frac{D}{m}\right)\dot{x} + \left(\frac{2k}{m}\right)x = 0$$

(or alternately

$$\frac{d^2x}{dt^2} + \left(\frac{D}{m}\right)\frac{dx}{dt} + \left(\frac{2k}{m}\right)x = 0 \ )$$

b.) This looks NASTY, but it is a solvable second-order differential equation.

7.) But what would our previous analysis look like if it was done it in terms of the mass's velocity?



a.) In that case, remembering that $x = \int v \, dt$, Newton's Second Law would yield:

$$\underline{\sum F:}$$

$$F_{spring1} + F_{spring2} + F_{damping} = ma$$

$$\Rightarrow \quad [-k(x)] + [-k(x)] + [-D(v)] = ma$$

$$\Rightarrow \quad (-2k)\int v\,dt + [-Dv] = m\dot{v}$$

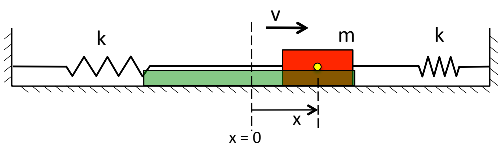
$$\Rightarrow \quad \dot{v} + \left(\frac{D}{m}\right)v + \left(\frac{2k}{m}\right)\int v\,dt = 0$$

(or alternately

$$\frac{dv}{dt} + \left(\frac{D}{m}\right)v + \left(\frac{2k}{m}\right)\int v\,dt = 0 \;)$$

b.) This is an equivalent equation, but in not nearly as solvable a form.

7.) The point here is that the solution to the question, "How does the velocity act in this situation?" is wrapped up in the solution of either:



$$\dot{v} + \left(\frac{D}{m}\right)v + \left(\frac{2k}{m}\right)\int v\,dt = 0 \quad \text{(which is } \frac{dv}{dt} + \left(\frac{D}{m}\right)v + \left(\frac{2k}{m}\right)\int v\,dt = 0\text{)}$$

or

$$\ddot{x} + \left(\frac{D}{m}\right)\dot{x} + \left(\frac{2k}{m}\right)x = 0 \quad \text{(which is } \frac{d^2x}{dt^2} + \left(\frac{D}{m}\right)\frac{dx}{dt} + \left(\frac{2k}{m}\right)x = 0\text{)}$$

WOULDN'T IT BE COOL IF WE COULD CIRCUMVENT ALL OF THIS MATH AND GET THE SOLUTION BY BEING CLEVER?

TO THAT END . . .

ELECTROMECHANICS

along with more Preliminary Information

So let's talk a little about electricity and magnetism and the devices designed to take advantage of the physics wrapped up in those topics.

1.) a VOLTAGE difference between two points:

a.) the voltage at a point is defined as the amount of *potential energy per unit charge* available at that point. It is a modified *potential energy* quantity.

b.) if you have a voltage *difference* between two points, an electric field (a modified force field) is set up that motivates charge to move.

c.) a 6 volts battery has a 6 volt *voltage difference* between its positive and negative terminals. If a wire connects its terminals, it will generate what is called a DC current. This is a current that flows in only one direction.

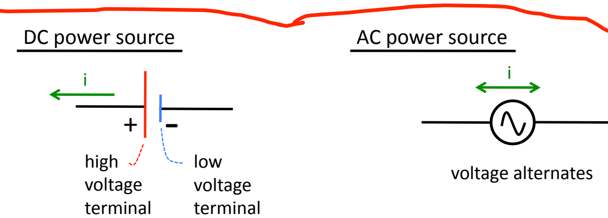
d.) there are power supplies (like your wall socket) that provide a voltage that *changes in magnitude and direction*. This produces what is called an (alternating, or) AC current.

2.) a CURRENT is formally defined as the amount of charge q that passes by a point *per unit time* Δt. (This is most easily visualized in a DC circuit.)

a.) Its symbol is "i." Mathematically, it is defined either as $\frac{\Delta q}{\Delta t}$ or $\frac{dq}{dt}$.

b.) As weird as this is going to seem, given what you know, the direction of a DC current is defined as the direction *positive charges* would move in a circuit, assuming they *could* move in a circuit. (In these circuits, current is assumed to flow from the high voltage terminal to the low voltage terminal.)

c.) Given the fact that AC voltages alternate back and forth, both charge flow and the direction of an AC current alternates back and forth.



3.) Resistors:

a.) Resistors are circuit elements that do two things:

i.) they dissipate energy (this is why they heat up when current flows through them, and.

ii.) they regulate current flow (for a given voltage, a large resistor will see a small current through its circuit; a small resistor will see a large current).

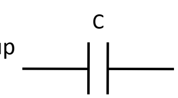
b.) Ohm's Law maintains that the *voltage difference* across a resistor is proportional to the *current through* the resistor. The proportionality constant is the resistance *R* of the resistor. Mathematically, Ohm's Law is written as:

$$V_R = iR \qquad \text{OR} \qquad V_R = \left(\frac{dq}{dt}\right)R$$

c.) Resistors act the same in AC and DC circuits. The circuit symbol for a resistor is shown to the right.

4.) Capacitors:

a.) Physically, a capacitors is a circuit element that is made up of an insulating material separating two insulated, metallic plates. Its symbol, shown to the right, reflects this make-up:



b.) In a DC circuit, positive charge accumulates on one plate electrostatically repulsing an equal amount of positive charge of the other plate leaving it electrically negative. The ratio of the amount of charge Q on one plate is proportional to the voltage V_c across the plates with the proportionality constant being the capacitor's capacitance C. Mathematically, this can be written as:

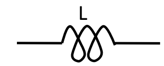
$$q = CV_c \quad \text{or} \quad C = \frac{q}{V_C}$$

Written a little differently, the voltage across a capacitor will equal:

$$V_C = \frac{1}{C}q$$

5.) Inductors:

a.) Physically, an inductor is a coil. Its symbol, somewhat inelegantly shown to the right, reflects this:



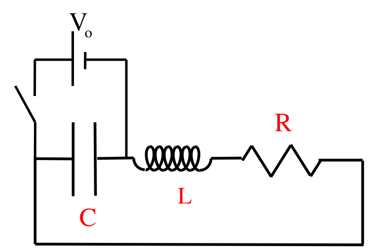
b.) In any circuit, a CHANGE of current di/dt through the coil will induce a voltage across the inductor. The size of the induced voltage V_L is proportional to the rate at which the current is changing di/dt. The proportionality constant between the two is called the inductance L of the inductor. Mathematically, this can be characterized as:

$$V_L = L\frac{di}{dt}$$

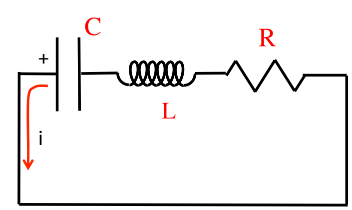
In terms of charge flow, this is the same as $V_L = L\frac{d^2q}{dt^2}$.

c.) As a side note, inductors, being made of wire, also have resistor-like resistance associated with them.

6.) An RLC circuit: Let's do something exotic and clever.



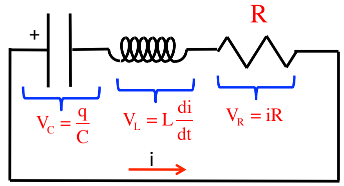
a.) By throwing the switch in the circuit to the right, the battery will place a voltage across the capacitor plates charging it up.



b.) By opening the switch, the battery will be removed and the capacitor will produce a current by discharging through the inductor and resistor.

c.) As the discharge happens, a voltage will develop across all three elements (remember, the charge on the plates produces the capacitor's voltage, the current through the resistor produces the resistor's voltage and the change in current produces the inductor's voltage).

d.) Kirchoff's Law states that the sum of the voltage changes around any circuit loop must equal to zero (think about it, the voltage difference between a point and itself will always be zero). That means we can write:



$$V_C + V_R + V_L = 0$$

$$\Rightarrow \quad \frac{q}{C} - iR + L\frac{di}{dt} = 0$$

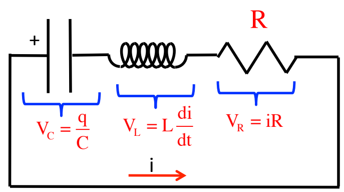
e.) On the surface, this may not look hopeful. If we write this out in *q*, where q is the charge on the capacitor, the rate at which charge is flowing through the circuit will be -dq/dt and we can write:

$$\frac{1}{C}q - (-\dot{q})R + L\ddot{q} = 0$$

$$\Rightarrow \quad \ddot{q} + \left(\frac{R}{L}\right)\dot{q} + \left(\frac{1}{LC}\right)q = 0$$

Why is i = - dq/dt? Ask your teacher! It has to do with the relationship between the the charge on the capacitor plates, defined as *q*, and the current (the *rate* at which charge flows in the circuit).

e.) So here is where the fun starts. This equation:



$$\ddot{q} + \left(\frac{R}{L}\right)\dot{q} + \left(\frac{1}{LC}\right)q = 0$$

Is the same differential equation that we determined for our spring system. That relationship looked like:

$$\ddot{x} + \left(\frac{D}{m}\right)\dot{x} + \left(\frac{2k}{m}\right)x = 0$$

f.) It appears that we might be able to build an electrical circuit with just the right size inductor, capacitor and resistor, so that the "motion" of charge in the circuit exactly mimic the "velocity" of the spring system we are interested in. All we have to do is determine the right parameters for the electrical system, charge up the capacitor, discharge it and look to see what the current does.

CLEVER, eh?

7.) Assignment 1: Knowing that $\ddot{q} + \left(\frac{R}{L}\right)\dot{q} + \left(\frac{1}{LC}\right)q = 0$ and $\ddot{x} + \left(\frac{D}{m}\right)\dot{x} + \left(\frac{2k}{m}\right)x = 0$:

1.) *Mass* is a measure of an object's *resistance to changing it's motion* (i.e., it's inertia).

a.) Which of the electrical components is the *mass* counterpart for our mechanical system?

b.) What does this tell you about the electrical element as it acts in this electrical circuit?

2.) *Damping* is a measure of external drag on a moving object in the sense that if the drag is great, the object's motion is limited greatly (and if it is small the object's motion is less limited).

a.) Which of the electrical components is the *damping* counterpart for our mechanical system?

b.) What does this tell you about the electrical element as it acts in this electrical circuit?

3.) *Velocity* is a measure of an object's *change of position* (i.e., the number of *meters per second* it covers at a given point in time).

a.) Which of the electrical components is the *velocity* counterpart for our mechanical system (this is a little obscure—be careful and BE COMPLETE)?

b.) What does this tell you about the electrical element as it acts in this electrical circuit?

4.) *Position* is a measure of a moving object's position at a given point in time.

a.) Which of the electrical components is the *position* counterpart for our mechanical system?

b.) What does this tell you about the electrical element as it acts in this electrical circuit?

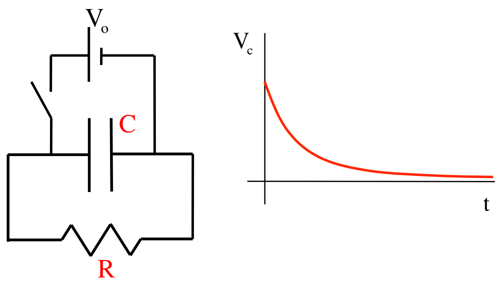
5.) The *spring constant* is a measure of the stiffness of the restoring force on the moving object.

a.) Which of the electrical components is the *spring constant* counterpart for our mechanical system?

b.) What does this tell you about the electrical element as it acts in this electrical circuit?

8.) So let's take a look at a few other electrical situations.

a.) A charged capacitor discharging through a single resistor has a discharge function that looks like the graph.



b.) Noting that because the cap is discharging, i = - dq/dt, it's differential equation is shown below.

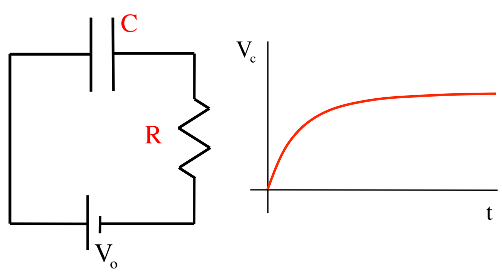
$$V_C + V_R = 0$$

$$\Rightarrow \frac{q}{C} - \left(-\frac{dq}{dt}\right)R = 0$$

$$\Rightarrow \dot{q} + \frac{1}{RC}q = 0$$

Note that the solution of this differential equation is an exponential, or $q = q_{max}e^{-t/RC}$

b.) A charging capacitor in series with a single resistor has a charging function that looks like the graph.



b.) Noting that because the cap is charging, i = +dq/dt, it's differential equation is shown below.

$$V_o \quad + \quad V_R \quad + \quad V_C \ = 0$$

$$\Rightarrow \frac{q_{max}}{C} - \left(\frac{dq(t)}{dt}\right)R - \frac{q(t)}{C} = 0$$

$$\Rightarrow \dot{q}(t) + \frac{1}{RC}q(t) = \frac{q_{max}}{RC}$$

Note that the solution of this differential equation is an exponential, or $q = q_{max}\left(1 - e^{-t/RC}\right)$

9.) Assignments 2: Use the information provided in Part 8a to identify a physical systems that might be modeled using an electrical system consisting of a discharging capacitor and resistor. You should draw a sketch of each system and write out the differential equations that define the system.

10.) Assignments 3: Use the information provided in Part 8b to identify a physical systems that might be modeled using an electrical system consisting of charging capacitor and resistor. You should draw a sketch of each system and write out the differential equations that define the system.